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**EQUATIONS FOR ISENTROPIC AND  
PLANE SHOCK FLOWS OF MIXTURES OF  
UNDISSOCIATED PLANETARY GASES**

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# EQUATIONS FOR ISENTROPIC AND PLANE SHOCK

## FLOWS OF MIXTURES OF UNDISSOCIATED

### PLANETARY GASES

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#### SUMMARY

Equations for aerodynamic quantities of interest in equilibrium isentropic one-dimensional flows and for properties across plane shock waves in arbitrary mixtures of planetary gases are derived under the assumption that the gases obey the perfect gas law. This assumption restricts the range of applicability of the equations to conditions for which the onset of chemical dissociation or ionization has not been reached although no requirement for the gases to be calorically perfect is imposed. Analytic functions used to specify the variation of specific heats with temperature are consistent with the assumption that molecular vibrations behave like harmonic oscillators and are independent of rotational degrees of freedom. Simplified equations giving approximate results over more limited ranges of the variables are also presented.

Sample comparisons of the aerodynamic behavior of a number of pure gases are made. It is shown that while some of the flow quantities characterizing aerodynamic flows are very sensitive to gas composition, the stagnation pressure coefficients for a number of gases differ by only a few percent from one gas to another.

#### INTRODUCTION

Considerable attention is now being directed toward the study of missions involving flight in the atmospheres of planets other than earth. The chemical compositions of the gases comprising these atmospheres are not known with any degree of precision at the present time. However, the presence or absence of certain constituents has been tentatively established for the near planets, and in some cases, abundances have been estimated. For example, evidence exists showing carbon dioxide to be present in the atmospheres of Mars and Venus in considerably larger amounts than found in the earth's atmosphere while methane and ammonia have been observed in the atmosphere of Jupiter. Various other gases such as nitrogen, argon, helium, and hydrogen have been considered as possible major constituents of planetary atmospheres. (See

reference 1 for a survey of information on planetary atmospheres.) Thus, a number of gas mixtures considerably different from air are being associated with other planet atmospheres.

In order to conduct analytical and experimental studies involving these proposed planetary gas mixtures it is necessary to understand the basic flows of these gases. Furthermore, if wind tunnels originally designed for operation with air are converted to operate with various planetary gas mixtures, methods will be needed for relating aerodynamic performance in arbitrary gas mixtures to that in air. Many of the proposed planetary gas mixtures are unlike air in that they cannot be treated as perfect gases even at low temperatures. Therefore, individual tables of flow information, such as those presented in reference 2, have been computed for gas mixtures of interest. It is certainly not practical to prepare tables of aerodynamic parameters for each and every proposed atmospheric gas model so that the need for analytic methods is clearly evident.

The present study is concerned with the development of equations for aerodynamic quantities of interest in equilibrium one-dimensional isentropic flows and adiabatic plane-shock flows for the ranges of conditions in which the gases considered obey the gas law  $p = \rho RT$ . The results are, therefore, generally applicable so long as the highest temperature in the flow does not exceed about 2500° K. This restricts the flow speed to the order of 2 to 4 km/sec or below for most gas mixtures. The equations are derived for arbitrary mixtures of monatomic gases such as helium and argon, linear molecule polyatomic gases such as nitrogen and carbon dioxide, and nonlinear molecule polyatomic gases not subject to internal rotation such as methane, ammonia, and water vapor. All of the polyatomic gases are admitted to be calorically imperfect with specific heats that vary with temperature. Within the framework of the assumptions results are obtained which are exact over the permissible ranges of variables. Simplified approximate results applicable over more limited ranges of the variables are also presented. The present results that apply to real undissociated air may be compared to those in reference 3. Unlike that study, however, specific heats are not restricted to vary in the first order only from their ideal gas values.

#### SYMBOLS

a	sound speed
A	area of flow cross section
$c_p$	specific heat per mole at constant pressure
$c_v$	specific heat per mole at constant volume
$C_{p_{stag}}$	pressure coefficient evaluated at stagnation conditions, $\frac{P_{t_2} - P_1}{q_1}$

e	Naperian logarithm base
E	energy per mole
$f_{\tau_p}, f_{\tau_T}, f_{\tau_\rho}$	fractional differences between $\tau_{\rho_1}/\rho_t$ , $\tau_{T_1}/T_t$ , $\tau_{\rho_1}/\rho_t$ and $\bar{\tau}_{p_1}/p_t$ , $\bar{\tau}_{T_1}/T_t$ , $\bar{\tau}_{\rho_1}/\rho_t$ , respectively
H	enthalpy per mole
m	molecular weight
M	Mach number
N	number of degrees of vibrational freedom
p	pressure
q	dynamic pressure, $\frac{\rho V^2}{2}$
R	gas constant, $\frac{\mathcal{R}}{m}$
$\mathcal{R}$	universal gas constant per mole
s	number of atoms in molecule
T	absolute temperature
$u(T)$	function of temperature defined by equation (8)
$v(T)$	function of temperature defined by equation (8)
V	speed
$w(T)$	function of temperature defined by equation (15)
x	mole fraction
$\beta$	shock-wave angle measured from downstream flow direction
$\gamma$	ratio of specific heats, $\frac{c_p}{c_v}$
$\delta$	angle of flow deflection across an oblique shock wave
$\eta$	function defined by equation (22)
$\theta$	molecular vibrational energy constant
$\rho$	mass density

$\tau( )$	ratio of ( ) for real gas to ( ) for ideal gas
$( )_v$	partial derivative at constant volume
$(\bar{ )}$	asymptotic value of quantity

### Subscripts

i	ideal-gas quantity
j,k,l	indices referring to monatomic gases, polyatomic linear molecule gases, and polyatomic nonlinear molecule gases not subject to internal rotation, respectively
n	index denoting degree of vibrational freedom
t	total conditions for gas brought isentropically to rest (wind-tunnel reservoir conditions)
air	air quantity
trans, rot, } vib, el }	translational, rotational, vibrational, and electronic energy modes, respectively
*	quantity evaluated at sonic conditions
1	conditions upstream of shock wave
2	conditions downstream of shock wave

### ANALYSIS

The analysis is divided into four sections: equations governing the thermodynamics of the gas mixtures are developed in the first, the second treats the one-dimensional isentropic flow situation, the third is devoted to plane shock flows, and the last presents simplified approximate forms of the equations. The equations presented only apply to mixtures of thermally perfect undissociated gases.

### Thermodynamic Properties

The thermodynamic properties of a mixture of gases can be calculated from the thermodynamic properties of the individual components or species. Thus, attention is first directed toward a general definition of the

necessary properties of an arbitrary pure gas. The ideas are then specialized to gases having specific types of molecular structure. Finally, equations for arbitrary mixtures of these gases are developed and a single equivalent gas is defined which has the exact thermodynamic properties of the actual gas mixture being considered.

General considerations.- The total energy possessed by a gas molecule can be written generally as

$$E = E_{\text{trans}} + E_{\text{int}}$$

where  $E_{\text{trans}}$  denotes the energy of translation of the molecule and  $E_{\text{int}}$  denotes the energy associated with the internal structure of the molecule. The internal energy  $E_{\text{int}}$  can arise from three sources:

1.  $E_{\text{rot}}$  rotation of the molecule about its mass center
2.  $E_{\text{vib}}$  vibration of the atoms in the molecule relative to each other
3.  $E_{\text{el}}$  excitation of the electrons within the molecule

This study is restricted to gases for which these three phenomena can be assumed, with good approximation, to be independent. Therefore,

$$E = E_{\text{trans}} + E_{\text{rot}} + E_{\text{vib}} + E_{\text{el}}$$

The translational degrees of freedom are fully excited at all temperatures and quantum mechanical considerations show the rotational degrees of freedom to be fully excited at temperatures above a few degrees Kelvin. Thus, values for  $E_{\text{trans}}$  and  $E_{\text{rot}}$  are readily obtained by using the law of equipartition of energy which states that each fully excited classical degree of freedom in a molecule contributes  $(1/2)RT$  to the specific energy. The energy associated with each degree of vibrational freedom in a molecule is determined from quantum mechanical theory by considering the vibrations to behave like harmonic oscillators. The energy in one degree of vibrational freedom is given by  $[RT(\theta/T)]/(e^{\theta/T} - 1)$  where  $\theta$  is the characteristic temperature of vibration. For a gas with  $N$  degrees of vibrational freedom

$$E_{\text{vib}} = RT \sum_{n=1}^N \frac{\theta_n/T}{e^{\theta_n/T} - 1}$$

Note that each fully excited degree of vibrational freedom ( $T \gg \theta_n$ ) behaves as two classical degrees of freedom by contributing  $RT$  to the specific

energy. The energy associated with electronic excitation is neglected in this study. At temperatures below those for molecular dissociation the electronic energy is small compared to the total energy.

The specific heat at constant volume is defined as

$$c_v \equiv \left( \frac{\partial E}{\partial T} \right)_v$$

so that

$$\frac{c_v}{R} = \frac{1}{R} \left( \frac{\partial E_{\text{trans}}}{\partial T} \right)_v + \frac{1}{R} \left( \frac{\partial E_{\text{rot}}}{\partial T} \right)_v + \frac{1}{R} \left( \frac{\partial E_{\text{vib}}}{\partial T} \right)_v$$

It is apparent from the above equation that each degree of translational and rotational freedom contributes 1/2 to the total  $c_v/R$ , independent of temperature, while the vibrational degrees of freedom each contribute between 0 and 1 to the total  $c_v/R$ , depending upon the temperature. Thus,  $c_v/R$  can be written in terms of a part independent of temperature  $(c_v/R)_i$  and a part due to vibrations  $(c_v/R)_{\text{vib}}$  which is dependent on temperature so that

$$\frac{c_v}{R} = \left( \frac{c_v}{R} \right)_i + \left( \frac{c_v}{R} \right)_{\text{vib}}$$

The relation between specific heats,  $c_p - c_v = R$ , is used to obtain

$$\frac{c_p}{R} = 1 + \left( \frac{c_v}{R} \right)_i + \left( \frac{c_v}{R} \right)_{\text{vib}}$$

An ideal-gas value of the ratio of specific heats is defined as

$$\gamma_i \equiv \frac{(c_p/R)_i}{(c_v/R)_i} = \frac{(c_v/R)_i + 1}{(c_v/R)_i} = 1 + \frac{1}{(c_v/R)_i}$$

so that, finally,

$$\left. \begin{aligned}
 \gamma_i &= 1 + \frac{1}{(c_v/R)_i} \\
 \frac{c_v}{R} &= \frac{1}{\gamma_i - 1} + \left(\frac{c_v}{R}\right)_{\text{vib}} \\
 \frac{c_p}{R} &= \frac{\gamma_i}{\gamma_i - 1} + \left(\frac{c_v}{R}\right)_{\text{vib}} \\
 \gamma &\equiv \frac{(c_p/R)}{(c_v/R)} = \frac{\gamma_i + (\gamma_i - 1)(c_v/R)_{\text{vib}}}{1 + (\gamma_i - 1)(c_v/R)_{\text{vib}}}
 \end{aligned} \right\} \quad (1)$$

The enthalpy is defined as

$$H \equiv R \int_0^T \left(\frac{c_p}{R}\right) dT \quad (2)$$

This definition implies that the enthalpy is zero when the gas is in molecular form at a temperature of absolute zero.

All of the thermodynamic properties needed in the analysis to follow have been written in terms of  $(c_v/R)_i$  and  $(c_v/R)_{\text{vib}}$ . Equations for each of these quantities will now be developed for each of three types of gases.

Monatomic gases.— These gases have no molecular structure since they are comprised of single atoms. Thus, they have no energy in either rotational or vibrational degrees of freedom. Their entire energy is included in the three degrees of translational freedom so that

$$\left. \begin{aligned}
 \left(\frac{c_v}{R}\right)_i &= \frac{3}{2} \\
 \left(\frac{c_v}{R}\right)_{\text{vib}} &= 0 \\
 \gamma_i &= 1 + \frac{1}{3/2} = \frac{5}{3}
 \end{aligned} \right\} \quad (3)$$

Polyatomic gases comprised of linear molecules.- The atoms comprising the molecules of these gases are arranged in a straight line. Each molecule possesses three degrees of translational freedom, two degrees of rotational freedom and one or more degrees of vibrational freedom depending upon the number of atoms in the molecule. These facts lead to the following results.

$$\left. \begin{aligned}
 \left(\frac{c_v}{R}\right)_i &= \frac{3}{2} + \frac{2}{2} = \frac{5}{2} \\
 \left(\frac{c_v}{R}\right)_{\text{vib}} &= \sum_{n=1}^{3s-5} \left(\frac{\theta_n}{T}\right)^2 \frac{e^{\theta_n/T}}{\left(e^{\theta_n/T} - 1\right)^2} \\
 \gamma_i &= 1 + \frac{1}{5/2} = \frac{7}{5}
 \end{aligned} \right\} \quad (4)$$

Polyatomic gases comprised of nonlinear molecules not subject to internal rotation.- The molecules of these gases differ from the molecules of the linear molecule gases in that all atoms are not arranged in a straight line. Under these circumstances, energy can be stored in three, rather than two, degrees of rotational freedom. Furthermore, the number of vibrational degrees of freedom is given by  $n = 3s-6$  instead of  $n = 3s-5$  as in the previous case.

$$\left. \begin{aligned}
 \left(\frac{c_v}{R}\right)_i &= \frac{3}{2} + \frac{3}{2} = 3 \\
 \left(\frac{c_v}{R}\right)_{\text{vib}} &= \sum_{n=1}^{3s-6} \left(\frac{\theta_n}{T}\right)^2 \frac{e^{\theta_n/T}}{\left(e^{\theta_n/T} - 1\right)^2} \\
 \gamma_i &= 1 + \frac{1}{3} = \frac{4}{3}
 \end{aligned} \right\} \quad (5)$$

Mixtures of nonreacting gases.- The thermodynamic properties of a mixture of nonreacting gases are given by the general relations of equations (1) and (2) provided the functions  $(c_v/R)_i$  and  $(c_v/R)_{\text{vib}}$  are suitably defined. Let

$$\left(\frac{c_v}{R}\right)_i \equiv \sum_j x_j \left(\frac{1}{\gamma_i - 1}\right)_j + \sum_k x_k \left(\frac{1}{\gamma_i - 1}\right)_k + \sum_l x_l \left(\frac{1}{\gamma_i - 1}\right)_l$$

and

$$\left(\frac{c_v}{R}\right)_{\text{vib}} \equiv \sum_j x_j \left[\left(\frac{c_v}{R}\right)_{\text{vib}}\right]_j + \sum_k x_k \left[\left(\frac{c_v}{R}\right)_{\text{vib}}\right]_k + \sum_l x_l \left[\left(\frac{c_v}{R}\right)_{\text{vib}}\right]_l$$

Observe that  $\left[\left(c_v/R\right)_{\text{vib}}\right]_j \equiv 0$  and that  $(\gamma_i)_j = 5/3$ ,  $(\gamma_i)_k = 7/5$ , and  $(\gamma_i)_l = 4/3$ . Substituting these values into the equations and letting  $(c_v/R)_{\text{vib}} \equiv u(T)$  gives

$$\left(\frac{c_v}{R}\right)_i = 1.5 \sum_j x_j + 2.5 \sum_k x_k + 3.0 \sum_l x_l$$

$$\left(\frac{c_v}{R}\right)_{\text{vib}} \equiv u(T) = \sum_k x_k u_k(T) + \sum_l x_l u_l(T)$$

where

$$u_k(T) = \sum_{n=1}^{3S-5} \left(\frac{\theta_{kn}}{T}\right)^2 \frac{e^{\theta_{kn}/T}}{\left(e^{\theta_{kn}/T} - 1\right)^2}$$

$$u_l(T) = \sum_{n=1}^{3S-6} \left(\frac{\theta_{ln}}{T}\right)^2 \frac{e^{\theta_{ln}/T}}{\left(e^{\theta_{ln}/T} - 1\right)^2}$$

(6)

The enthalpy of the gas mixture, from equation (2), is

$$H = R \int_0^T \left( \frac{c_p}{R} \right) dT = R \int_0^T \left[ \frac{\gamma_i}{\gamma_i - 1} + u(T) \right] dT$$

$$H = \left( \frac{\gamma_i}{\gamma_i - 1} \right) RT + R \int_0^T u(T) dT$$

Let

$$v(T) \equiv \frac{1}{T} \int_0^T u(T) dT = \sum_k x_k v_k(T) + \sum_l x_l v_l(T)$$

where

$$v_k(T) = \sum_{n=1}^{3S-5} \frac{\theta_{kn}/T}{e^{\theta_{kn}/T} - 1} \quad \text{and} \quad v_l(T) = \sum_{n=1}^{3S-6} \frac{\theta_{ln}/T}{e^{\theta_{ln}/T} - 1}$$

then

$$\frac{H}{RT} = \frac{\gamma_i}{\gamma_i - 1} + v(T) \quad (7)$$

Note that the function  $v(T)$  is nothing more than  $E_{vib}/RT$ .

The gas constant  $R$  for the mixture is given by

$$R = \frac{\mathcal{R}}{m}$$

and the molecular weight  $m$  is given by

$$m = \sum_j x_j m_j + \sum_k x_k m_k + \sum_l x_l m_l$$

Summary of thermodynamic relations for gas mixtures.-

$$\left(\frac{c_v}{R}\right)_i = 1.5 \sum_j x_j + 2.5 \sum_k x_k + 3.0 \sum_l x_l$$

$$\gamma_i = 1 + \frac{1}{(c_v/R)_i}$$

$$\gamma = \frac{\gamma_i + (\gamma_i - 1)u(T)}{1 + (\gamma_i - 1)u(T)}$$

$$\frac{H}{RT} = \frac{\gamma_i}{\gamma_i - 1} + v(T)$$

$$m = \sum_j x_j m_j + \sum_k x_k m_k + \sum_l x_l m_l$$

$$R = \frac{\mathcal{R}}{m}$$

$$u(T) = \left(\frac{c_v}{R}\right)_{\text{vib}} = \sum_k x_k u_k(T) + \sum_l x_l u_l(T)$$

$$v(T) = \frac{E_{\text{vib}}}{RT} = \sum_k x_k v_k(T) + \sum_l x_l v_l(T)$$

$$\frac{c_v}{R} = \left(\frac{c_v}{R}\right)_i + \left(\frac{c_v}{R}\right)_{\text{vib}}$$

$$u_k(T) = \sum_{n=1}^{3S-5} \left(\frac{\theta_{kn}}{T}\right)^2 \frac{e^{\theta_{kn}/T}}{\left(e^{\theta_{kn}/T} - 1\right)^2}; \quad u_l(T) = \sum_{n=1}^{3S-6} \left(\frac{\theta_{ln}}{T}\right)^2 \frac{e^{\theta_{ln}/T}}{\left(e^{\theta_{ln}/T} - 1\right)^2}$$

$$v_k(T) = \sum_{n=1}^{3S-5} \frac{\theta_{kn}/T}{e^{\theta_{kn}/T} - 1}; \quad v_l(T) = \sum_{n=1}^{3S-6} \frac{\theta_{ln}/T}{e^{\theta_{ln}/T} - 1}$$

(8)

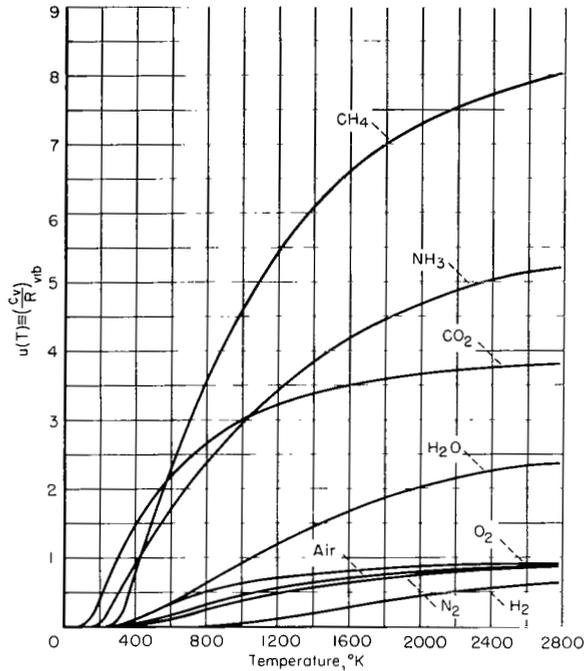


Figure 1.- Variation of the function  $u(T)$  with temperature for several gases.

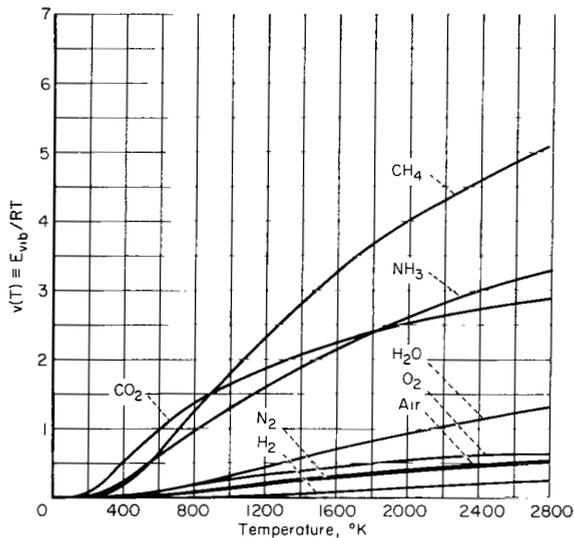


Figure 2.- Variation of the function  $v(T)$  with temperature for several gases.

The functions  $u(T)$  and  $v(T)$  have been evaluated for a number of the gases considered in the compositions of planetary atmospheres. The results of these calculations are presented in figures 1 and 2. Values of the characteristic temperatures of vibration for a number of gases are given in table I. With the exception of air, the values given are consistent with those tabulated in references 4 and 5. The characteristic temperature used for air is that suggested in reference 3.

### One-Dimensional Isentropic Flows

The equations for quantities of interest in one-dimensional isentropic flows of nonreacting gas mixtures are developed in this section. The results of the preceding section showed that an equivalent gas having the thermodynamic properties of a specified mixture of gases could be defined. The equations of this section will be derived in terms of the thermodynamic functions of this equivalent gas. The thermodynamically equivalent gas has a ratio of specific heats  $\gamma$  that is variable with temperature at high temperatures but which approaches a constant value at low temperatures. Flow properties for the gas mixtures computed using this constant low temperature value of specific heat ratio  $\gamma_1$  and the ideal gas equations such as found in reference 6 are called ideal-gas values. It has been found that equations for the ratio of quantities for the real gas to the same quantities for the ideal gas take particularly simple forms. Thus, equations are presented for these ratios. Simplified forms of some of these equations that give results accurate to within specified limits over limited ranges of the variables are derived in a later section.

The equations governing the four unknowns  $p$ ,  $\rho$ ,  $V$ , and  $T$  can be written as follows:

$$\left. \begin{array}{l}
 \text{Mass} \qquad \qquad \qquad d(\rho VA) = 0 \\
 \text{Entropy} \qquad \qquad \qquad \frac{d\rho}{\rho} - \frac{c_v}{R} \frac{dT}{T} = 0 \\
 \text{Energy} \qquad \qquad \qquad V dV + dH = 0 \\
 \text{State} \qquad \qquad \qquad p = \rho RT ; \quad H = H(T) ; \quad \frac{c_v}{R} = \frac{c_v}{R} (T)
 \end{array} \right\} \quad (9)$$

The enthalpy and specific heat functions are given explicitly by equations (8).

A relation between flow speed and temperature can be obtained directly by integrating the energy equation. The result is

$$V_1^2 = 2RT_1 \frac{\gamma_i}{\gamma_i - 1} \left\{ \left( \frac{T_t}{T_1} - 1 \right) + \frac{\gamma_i - 1}{\gamma_i} \left[ \frac{T_t}{T_1} v(T_t) - v(T_1) \right] \right\} \quad (10)$$

Both sides of equation (10) can be divided by the square of the sound speed to obtain the following relation for Mach number

$$M_1^2 = \frac{2}{(\gamma_i - 1)} \frac{\gamma_i}{\gamma_i} \left\{ \left( \frac{T_t}{T_1} - 1 \right) + \frac{\gamma_i - 1}{\gamma_i} \left[ \frac{T_t}{T_1} v(T_t) - v(T_1) \right] \right\} \quad (11)$$

It is recognized that by rearranging terms in equation (11) and by adding unity to both sides the following result can be obtained

$$\begin{aligned}
 \tau_{T_1/T_t} &\equiv \frac{T_1/T_t}{(T_1/T_t)_i} = \frac{T_1/T_t}{\left( 1 + \frac{\gamma_i - 1}{2} M_1^2 \right)^{-1}} \\
 &= \frac{\gamma_i}{\gamma_i} \left\{ 1 + \frac{\gamma_i - 1}{\gamma_i} v(T_t) + \frac{T_1}{T_t} \left[ \left( 1 - \frac{\gamma_i}{\gamma_i} \right) + \frac{\gamma_i - 1}{\gamma_i} v(T_1) \right] \right\} \quad (12)
 \end{aligned}$$

where use has been made of the fact that

$$\left(\frac{T_1}{T_t}\right)_i = \left(1 + \frac{\gamma_i - 1}{2} M_1^2\right)^{-1}$$

This function and all other ideal-gas functions employed in this analysis are given in reference 6. Equation (12) gives the correction to be applied to the ideal-gas results to account for the real-gas caloric imperfection. This correction is written in terms of  $T_1/T_t$  and either  $T_1$  or  $T_t$  but it can be related to flow Mach number or speed in the following manner. In a flight application the static temperature  $T_1$  would be known and in a wind-tunnel application the total temperature  $T_t$  would be known so that the independent variable in equation (12) is the ratio  $T_1/T_t$ . By use of the definition of  $\tau_{T_1/T_t}$ , the following equations for Mach number and speed can be obtained

$$M_1^2 = \frac{2}{\gamma_i - 1} \left( \frac{\tau_{T_1/T_t}}{T_1/T_t} - 1 \right) \quad (13)$$

$$V_1^2 = \frac{2\gamma_i R T_1}{\gamma_i - 1} \left( \frac{\tau_{T_1/T_t}}{T_1/T_t} - 1 \right) \quad (14)$$

The right-hand sides of equations (13) and (14) are functions only of  $T_1/T_t$  once either  $T_1$  or  $T_t$  is specified. Now that flow speed and Mach number have been related to temperature it is sufficient to determine each of the remaining isentropic flow quantities in terms of the static-to-total temperature ratio  $T_1/T_t$  and either static or total temperature.

An equation for the real-gas correction to static-to-total density ratio is obtained by integrating the entropy equation. The result is

$$\frac{\rho_1}{\rho_t} = \left(\frac{T_1}{T_t}\right)^{\frac{1}{\gamma_i - 1}} \exp \int_{T_t}^{T_1} u(T) \frac{dT}{T}$$

Now, recognizing that

$$\left(\frac{\rho_1}{\rho_t}\right)_i = \left(\frac{T_1}{T_t}\right)_i^{\frac{1}{\gamma_i - 1}}$$

and defining

$$\tau_{\rho_1/\rho_t} \equiv \frac{\rho_1/\rho_t}{(\rho_1/\rho_t)_i}$$

allows the real-gas correction to the density ratio to be written

$$\tau_{\rho_1/\rho_t} = \left( \tau_{T_1/T_t} \right)^{\frac{1}{\gamma_i - 1}} \exp \int_{T_t}^{T_1} u(T) \frac{dT}{T}$$

The integral in the equation above can be evaluated analytically. The result is

$$\int_{T_t}^{T_1} u(T) \frac{dT}{T} = w(T_1) - w(T_t)$$

where

$$\left. \begin{aligned} w(T) &= \sum_k x_k w_k(T) + \sum_l x_l w_l(T) \\ w_k(T) &= \sum_{n=1}^{3S-5} \left[ \frac{\theta_{kn}}{T} \frac{e^{\theta_{kn}/T}}{(e^{\theta_{kn}/T} - 1)} - \ln \left( e^{\theta_{kn}/T} - 1 \right) \right] \\ w_l(T) &= \sum_{n=1}^{3S-6} \left[ \frac{\theta_{ln}}{T} \frac{e^{\theta_{ln}/T}}{(e^{\theta_{ln}/T} - 1)} - \ln \left( e^{\theta_{ln}/T} - 1 \right) \right] \end{aligned} \right\} \quad (15)$$

The functions  $w_k(T)$  and  $w_l(T)$  need be evaluated only once for each pure gas after which the function  $w(T)$  for any possible mixture of the pure gases can be formed. For convenience, the  $w(T)$  function has been evaluated for a

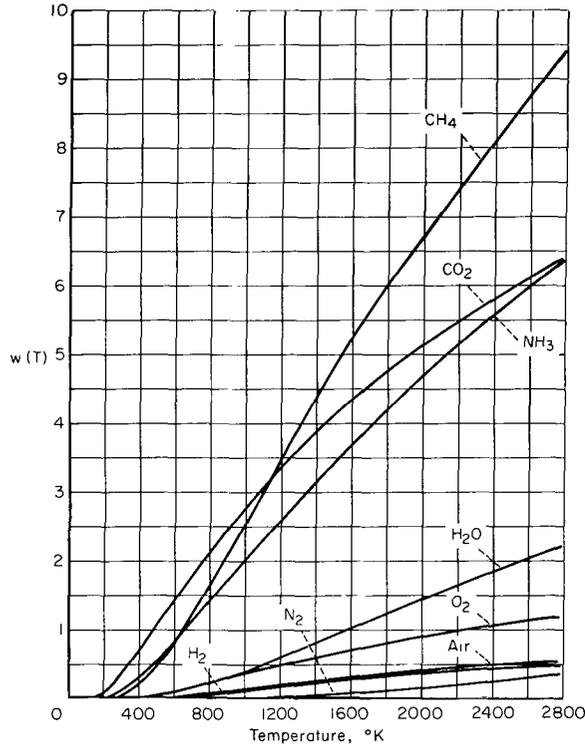


Figure 3.- Variation of the function  $w(T)$  with temperature for several gases.

number of pure gases and the results are presented in figure 3. Finally, the real-gas correction to the static-to-total density ratio is written

$$\tau_{\rho_1/\rho_t} = \left( \tau_{T_1/T_t} \right)^{\frac{\gamma_i}{\gamma_i - 1}} e^{[w(T_1) - w(T_t)]} \quad (16)$$

The real-gas correction to the static-to-total pressure ratio is obtained in terms of the temperature and density corrections by using the equation of state. The result is

$$\begin{aligned} \tau_{p_1/p_t} &\equiv \frac{p_1/p_t}{(p_1/p_t)_i} = \left( \tau_{T_1/T_t} \right) \left( \tau_{\rho_1/\rho_t} \right) \\ &= \left( \tau_{T_1/T_t} \right)^{\frac{\gamma_i}{\gamma_i - 1}} e^{[w(T_1) - w(T_t)]} \end{aligned} \quad (17)$$

The real-gas correction to the dynamic-to-total pressure ratio is obtained as follows

$$\begin{aligned} \tau_{q_1/p_t} &\equiv \frac{q_1/p_t}{(q_1/p_t)_i} = \frac{\rho_1 V_1^2/p_t}{(\rho_1 V_1^2/p_t)_i} = \frac{\gamma_1 (p_1/p_t) M_1^2}{\gamma_i (p_1/p_t)_i (M_1^2)_i} \\ \tau_{q_1/p_t} &= \frac{\gamma_1}{\gamma_i} \tau_{p_1/p_t} \end{aligned} \quad (18)$$

An equation for the real-gas correction to the flow area ratio  $A/A_*$  is developed from the mass continuity equation. Integrating the first of equations (9) gives

$$\rho_1 V_1 A = \rho_* V_* A_*$$

where the \* denotes conditions evaluated at the sonic throat. Then,

$$\frac{A}{A_*} = \frac{\rho_*/\rho_t}{\rho_1/\rho_t} \frac{a_*}{a} \frac{1}{M_1} = \frac{\rho_*/\rho_t}{\rho_1/\rho_t} \sqrt{\frac{\gamma_* T_*}{\gamma_1 T_1}} \frac{1}{M_1}$$

but

$$\left(\frac{A}{A_*}\right)_i = \frac{(\rho_*/\rho_t)_i}{(\rho_1/\rho_t)_i} \sqrt{\frac{(T_*/T_t)_i}{(T_1/T_t)_i}} \frac{1}{M_1}$$

so that

$$\tau_{A/A_*} \equiv \frac{A/A_*}{(A/A_*)_i} = \frac{\tau_{\rho_*/\rho_t}}{\tau_{\rho_1/\rho_t}} \sqrt{\frac{\tau_{T_*/T_t} \gamma_*}{\tau_{T_1/T_t} \gamma_1}} \quad (19)$$

Real-gas corrections to other flow quantities can be formed using combinations of those presented.

### Plane Shock Flows

The equations derived in this section relate flow quantities on either side of plane shock waves. As in the preceding section, the equations are cast in terms of the thermodynamic functions of the equivalent single gas having the exact properties of any specified mixture of gases.

The equations governing the four unknowns  $p_2$ ,  $\rho_2$ ,  $V_2$ , and  $T_2$  behind normal shock waves can be written as follows:

$$\left. \begin{array}{l} \text{Mass} \\ \text{Momentum} \\ \text{Energy} \\ \text{State} \end{array} \right\} \begin{array}{l} \rho_1 V_1 = \rho_2 V_2 \\ p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \\ H_1 + \frac{1}{2} V_1^2 = H_2 + \frac{1}{2} V_2^2 \\ p = \rho RT ; \quad H = H(T) \end{array} \quad (20)$$

The enthalpy function is given explicitly by equations (8).

A number of approaches to the derivation of the shock relations are valid although in some the algebra becomes cumbersome. It will be shown that the density and pressure relations contain terms involving the static temperature behind the shock so a logical approach would be to first derive the relation for static temperature ratio across a shock. In the interest of simplicity, however, the density and pressure equations are derived first and then are combined to obtain the temperature equation.

Manipulation of equations (20) results in the following quadratic equation for the density ratio  $\rho_1/\rho_2$

$$\begin{aligned} \frac{\gamma_i + 1}{2} \gamma_1 M_1^2 \left( \frac{\rho_1}{\rho_2} \right)^2 - \gamma_i (1 + \gamma_1 M_1^2) \left( \frac{\rho_1}{\rho_2} \right) + \gamma_i \\ + \frac{\gamma_1 M_1^2}{2} (\gamma_i - 1) - (\gamma_i - 1) \frac{T_2}{T_1} \left[ v(T_2) - \frac{T_1}{T_2} v(T_1) \right] = 0 \end{aligned}$$

This equation has the solution

$$\frac{\rho_1}{\rho_2} = \frac{\gamma_i (1 + \gamma_1 M_1^2) - \sqrt{(\gamma_1 M_1^2 - \gamma_i)^2 + 2(\gamma_i^2 - 1)\gamma_1 M_1^2 \frac{T_2}{T_1} \left[ v(T_2) - \frac{T_1}{T_2} v(T_1) \right]}}{\gamma_1 M_1^2 (\gamma_i + 1)} \quad (21)$$

Let

$$\eta \equiv (\gamma_1 M_1^2 - \gamma_i)^2 + 2(\gamma_i^2 - 1)\gamma_1 M_1^2 \frac{T_2}{T_1} \left[ v(T_2) - \frac{T_1}{T_2} v(T_1) \right] \quad (22)$$

and note that

$$\left( \frac{\rho_2}{\rho_1} \right)_i = \frac{(1 + \gamma_i) M_1^2}{2 + (\gamma_i - 1) M_1^2}$$

then

$$\tau_{\rho_2/\rho_1} \equiv \frac{\rho_2/\rho_1}{(\rho_2/\rho_1)_i} = \frac{2\gamma_i + (\gamma_i - 1)\gamma_1 M_1^2}{\gamma_i (1 + \gamma_1 M_1^2) - \sqrt{\eta}} \quad (23)$$

The equation for the static pressure ratio  $p_2/p_1$  is obtained by first noting that the momentum equation can be written

$$\frac{p_2}{p_1} = 1 + \gamma_1 M_1^2 \left( 1 - \frac{\rho_1}{\rho_2} \right) \quad (24)$$

Substitution of equation (21) for  $\rho_1/\rho_2$  into equation (24) gives

$$\frac{p_2}{p_1} = \frac{(1 + \gamma_1 M_1^2) + \sqrt{\eta}}{(\gamma_1 + 1)}$$

Now

$$\left( \frac{p_2}{p_1} \right)_i = \frac{2\gamma_1 M_1^2 - (\gamma_1 - 1)}{(\gamma_1 + 1)}$$

so that

$$\tau_{p_2/p_1} \equiv \frac{p_2/p_1}{(p_2/p_1)_i} = \frac{(1 + \gamma_1 M_1^2) + \sqrt{\eta}}{2\gamma_1 M_1^2 - (\gamma_1 - 1)} \quad (25)$$

The real-gas correction to the static temperature ratio is obtained by using the thermal equation of state and the preceding equations for density and pressure. The result is

$$\tau_{T_2/T_1} \equiv \frac{T_2/T_1}{(T_2/T_1)_i} = \frac{\gamma_1 (1 + \gamma_1 M_1^2)^2 - \eta + (\gamma_1 - 1)(1 + \gamma_1 M_1^2)\sqrt{\eta}}{\gamma_1 [2\gamma_1 M_1^2 - (\gamma_1 - 1)] [(\gamma_1 - 1)M_1^2 + 2]} \quad (26)$$

An iterative procedure is required to solve equation (26) for the correction to the static temperature ratio since the function  $\eta$  contains the unknown  $T_2$ . A simple, rapidly converging, method requires writing  $T_2$  in terms of the correction factor as follows:

$$T_2 = \tau_{T_2/T_1} \left( \frac{T_2}{T_1} \right)_i T_1$$

Then, set  $\tau_{T_2/T_1}$  equal to unity to obtain a first estimate of  $T_2$ . This value of  $T_2$  can be used to obtain a first estimate of  $\eta$  which, in turn, allows the calculation of  $\tau_{T_2/T_1}$ . The new value of  $\tau_{T_2/T_1}$  is then used to

provide a second estimate of  $T_2$ . The iteration is complete when the difference between successive values of  $\tau_{T_2/T_1}$  is within acceptable limits. It is shown in the section on approximations to the equations that the iteration can be avoided completely if results accurate to within several percent are acceptable.

The real-gas correction to the ratio of flow speeds across a normal shock is developed from the mass, momentum, and state equations.

$$\frac{V_2}{V_1} = \frac{1}{\gamma_1 M_1^2} \left( 1 + \gamma_1 M_1^2 - \frac{p_2}{p_1} \right) \quad (27)$$

The ideal-gas equation for  $(V_2/V_1)_i$  is

$$\left( \frac{V_2}{V_1} \right)_i = \frac{2 + M_1^2(\gamma_i - 1)}{M_1^2(\gamma_i + 1)} \quad (28)$$

Combining equations (27) and (28) and substituting a previous relation for  $p_2/p_1$  gives

$$\tau_{V_2/V_1} \equiv \frac{V_2/V_1}{(V_2/V_1)_i} = \frac{\gamma_i(1 + \gamma_1 M_1^2) - \sqrt{\eta}}{2\gamma_1 + \gamma_1 M_1^2(\gamma_i - 1)} \quad (29)$$

All of the remaining normal shock quantities can be developed from the results given above. The real-gas correction to the sound speed is obtained as follows:

$$\frac{a_2}{a_1} = \sqrt{\frac{\gamma_2 T_2}{\gamma_1 T_1}}$$

$$\left( \frac{a_2}{a_1} \right)_i = \sqrt{\left( \frac{T_2}{T_1} \right)_i}$$

so

$$\tau_{a_2/a_1} \equiv \frac{a_2/a_1}{(a_2/a_1)_i} = \sqrt{\tau_{T_2/T_1} \frac{\gamma_2}{\gamma_1}} \quad (30)$$

The Mach number ratio is given by inspection

$$\tau_{M_2/M_1} \equiv \frac{M_2/M_1}{(M_2/M_1)_i} = \frac{\tau_{V_2/V_1}}{\tau_{a_2/a_1}} = \frac{1}{(\tau_{\rho_2/\rho_1})(\tau_{a_2/a_1})} \quad (31)$$

The ratios of total conditions across normal shocks are given by the identities:

$$\left. \begin{aligned} \tau_{\rho_{t_2}/\rho_{t_1}} &\equiv \frac{\rho_{t_2}/\rho_{t_1}}{(\rho_{t_2}/\rho_{t_1})_i} = \frac{(\tau_{\rho_2/\rho_1})(\tau_{\rho_1/\rho_t})}{\tau_{\rho_2/\rho_t}} \\ \tau_{p_{t_2}/p_{t_1}} &\equiv \frac{p_{t_2}/p_{t_1}}{(p_{t_2}/p_{t_1})_i} = \tau_{\rho_{t_2}/\rho_{t_1}} \end{aligned} \right\} \quad (32)$$

The latter of equations (32) is obtained by making use of the fact that total temperature is conserved across the shock wave.

The equations developed for normal shock waves are also applicable to the study of oblique shock flows. Values of the ratios  $\tau_{\rho_2/\rho_1}$ ,  $\tau_{p_2/p_1}$ ,  $\tau_{T_2/T_1}$ , and  $\tau_{a_2/a_1}$  for oblique shock waves are given by equations (23), (25), (26), and (30), respectively, provided  $M_1 \sin \beta$  is used instead of  $M_1$  and the static temperature upstream of the oblique shock wave is the same as upstream of the normal shock wave. The flow turning angle  $\delta$  is related to the shock wave angle  $\beta$  by the following equation obtained from consideration of mass conservation

$$\frac{\tan(\beta - \delta)}{\tan \beta} = \frac{\rho_1}{\rho_2} = \frac{1}{\tau_{\rho_2/\rho_1} \left(\frac{\rho_2}{\rho_1}\right)_i} = \frac{2 + (\gamma_i - 1)M_1^2 \sin^2 \beta}{\tau_{\rho_2/\rho_1} (\gamma_i + 1)M_1^2 \sin^2 \beta} \quad (33)$$

The Mach number immediately downstream of an oblique shock wave can be determined from

$$\frac{M_2}{M_1} = \frac{\rho_1}{\rho_2} \frac{a_1}{a_2} \frac{1}{\sin(\beta - \delta)} = \frac{(T_1/T_2)_i (\rho_1/\rho_2)_i}{\left(\tau_{\rho_2/\rho_1}\right) \left(\tau_{a_2/a_1}\right) \sin(\beta - \delta)} \quad (34)$$

In the latter two equations the density, temperature, and sound-speed ratios are those corresponding to the upstream Mach number  $M_1$  and the shock wave angle  $\beta$ . When  $M_2$  is determined from equation (34), the remaining normal shock expressions, equations (32), can be applied to oblique shock flows.

### Approximate Results

Simplifications to a number of the equations of the preceding sections are suggested by a cursory analysis. It will be shown for isentropic flows that substantial simplifications can be made, without great loss in accuracy, when the static temperature is relatively low (room temperature or less). Fortunately, this condition exists in a number of practical problems. For example, in most supersonic wind-tunnel applications the stagnation temperature is so limited that the test section static temperature is nearly always of the order of room temperature or less. Also, in the flight application, the static temperature of the ambient atmosphere at altitudes sufficiently high for practical supersonic flight is generally low. It will also be shown that the equations for plane shock-wave flows can be simplified by imposing the additional condition that  $V_1^2/RT_1 \gg 1$  which implies the thermal energy of the gas to be much less than the kinetic energy of the flow. In addition to the simplifications, some rules to indicate the ranges of applicability of the approximate equations also will be derived.

Isentropic flows.- Consider first equation (12) for the real-gas correction to the static-to-total temperature ratio. When the static temperature is low then  $v(T_1) \rightarrow 0$ . For these conditions equation (12) takes the form

$$\tau_{T_1/T_t} \approx \frac{\gamma_i}{\gamma_1} \left[ 1 + \frac{\gamma_i - 1}{\gamma_i} v(T_t) \right] \quad (35)$$

A significant result deduced from this equation is the fact that when the static temperature is low it ceases to be a variable in the problem. This is made evident by recalling that the value of  $\gamma_1$  approaches the value of  $\gamma_i$  as the static temperature is lowered. Under these circumstances  $\gamma_i/\gamma_1 \rightarrow 1$  and the correction to the temperature ratio is a constant for a constant total temperature flow such as found in a wind tunnel. The value approached by  $\tau_{T_1/T_t}$  asymptotically as  $T_1 \rightarrow 0$  while  $T_t$  is held constant is given by

$$\bar{\tau}_{T_1/T_t} = 1 + \frac{\gamma_i - 1}{\gamma_i} v(T_t) \quad (36)$$

Substitution of equation (36) into equation (35) gives

$$\tau_{T_1/T_t} \approx \frac{\gamma_i}{\gamma_1} \bar{\tau}_{T_1/T_t} \quad (37)$$

An estimate of the error introduced by using equation (37) for  $\tau_{T_1/T_t}$  in place of the more complicated, but exact, equation (12) can now be made. Define

$$f_{\tau_T} \equiv \frac{\tau_{T_1/T_t} - \bar{\tau}_{T_1/T_t}}{\bar{\tau}_{T_1/T_t}}$$

as being the fractional error. Substituting from equation (37) gives

$$f_{\tau_T} = \frac{\gamma_i}{\gamma_1} - 1$$

Solving for  $\gamma_1$  in terms of the fractional error leads to

$$\gamma_1 = \frac{\gamma_i}{1 + f_{\tau_T}} \quad (38)$$

Equation (38) gives the value of  $\gamma_1$  corresponding to a given fractional error in  $\tau_{T_1/T_t}$ . This value of  $\gamma_1$  corresponds to a specific value of static temperature which can be determined from the equation relating  $\gamma_1$  to temperature (eqs. (8)). Once static temperature is determined then equations (13) and (14) give the flow Mach number and flow speed, respectively, above which equation (37) can be used for calculating  $\tau_{T_1/T_t}$  without incurring an error larger than  $f_{\tau_T}$ . Results of sample calculations illustrating the real-gas temperature ratio correction for pure carbon dioxide are presented

in figure 4. The curves depicting exact results were obtained from equations (12) and (13). The approximate results given by equations (37) and (13) are shown to agree closely with the exact curves for Mach numbers where the static

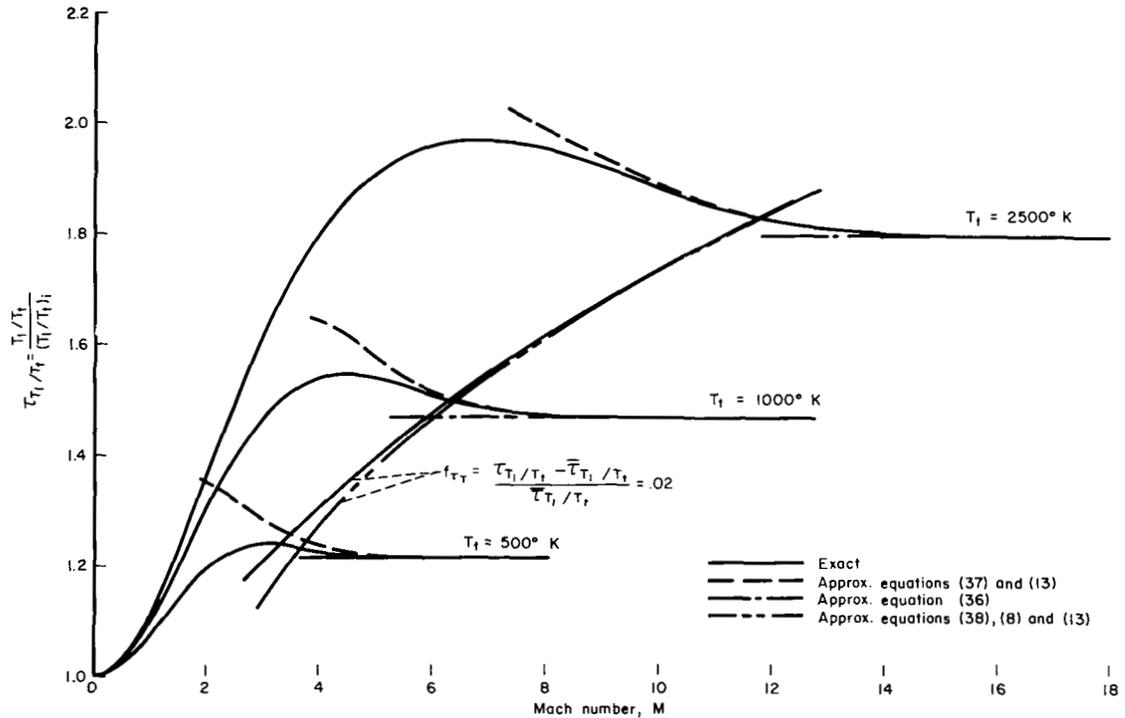


Figure 4.- Comparisons of exact and approximate results for the real gas correction to the static-to-total temperature ratio in an isentropic flow of carbon dioxide.

temperature is low. The asymptotic result given simply by equation (36) is shown to be essentially exact. The predicted and the exact lines of constant percent deviation from the asymptotic values of  $\tau_{T_1}/\tau_t$  are also shown. Note that the theory is conservative in predicting the Mach number for which approximate results deviate a specified percentage from the exact results. The same general procedure used to estimate the range of applicability of the simplified equation for the real-gas correction to the static-to-total temperature ratio can be used to provide similar estimates for the corrections to the other isentropic flow parameters.

In the case of the real-gas correction to the static-to-total density ratio, it is observed that the function  $e^{w(T_1)}$  in equation (16) approaches unity as the static temperature  $T_1$  becomes low so that it is convenient to define

$$\bar{\tau}_{\rho_1/\rho_t} \equiv \left( \bar{\tau}_{T_1/T_t} \right)^{\frac{1}{\gamma_1 - 1}} e^{-w(T_t)} \quad (39)$$

Equation (39) gives the value approached by  $\tau_{\rho_1/\rho_t}$  asymptotically as  $T_1$  becomes progressively lower while the total temperature is held constant. Using equation (39), equation (16) can be rewritten as follows

$$\tau_{\rho_1/\rho_t} = \left( \frac{\tau_{T_1/T_t}}{\bar{\tau}_{T_1/T_t}} \right)^{\frac{1}{\gamma_i - 1}} \bar{\tau}_{\rho_1/\rho_t} e^{w(T_1)} \quad (40)$$

An estimate of the flow Mach number for which a given fractional error is incurred by using the simpler equation, equation (39), in place of the exact result of equation (40) can now be made. The approximation for  $\tau_{T_1/T_t}$  given by equation (37) is substituted into equation (40) and the term  $e^{w(T_1)}$  is dropped so that

$$\tau_{\rho_1/\rho_t} \approx \left( \frac{\gamma_i}{\gamma_1} \right)^{\frac{1}{\gamma_i - 1}} \bar{\tau}_{\rho_1/\rho_t} \quad (41)$$

Now, write

$$\left( \frac{\gamma_i}{\gamma_1} \right)^{\frac{1}{\gamma_i - 1}} \equiv \left[ 1 + \left( \frac{\gamma_i}{\gamma_1} - 1 \right) \right]^{\frac{1}{\gamma_i - 1}}$$

Expanding the right side in a Taylor series and dropping higher order terms gives

$$\left( \frac{\gamma_i}{\gamma_1} \right)^{\frac{1}{\gamma_i - 1}} \approx 1 + \frac{1}{\gamma_i - 1} \left( \frac{\gamma_i}{\gamma_1} - 1 \right) \quad (42)$$

Define the fractional error as

$$f_{\tau\rho} \equiv \frac{\tau_{\rho_1/\rho_t} - \bar{\tau}_{\rho_1/\rho_t}}{\bar{\tau}_{\rho_1/\rho_t}}$$

and substitute equations (41) and (42) into this definition to get

$$f_{\tau\rho} = \frac{1}{\gamma_i - 1} \left( \frac{\gamma_i}{\gamma_1} - 1 \right)$$

Solving for  $\gamma_1$  gives

$$\gamma_1 = \frac{\gamma_i}{1 + (\gamma_i - 1)f_{\tau\rho}} \quad (43)$$

Equation (43) gives the value of specific heat ratio corresponding to a given fractional error in  $\tau_{p_1}/\rho_t$  resulting from using the simplified result of equation (39). By methods discussed previously, the value of  $\gamma_1$  given by equation (43) can be related to static temperature and, in turn, to either Mach number or flow speed.

An approach similar to that used above gives the following ratio of real-gas correction to the static-to-total pressure that is approached as the static temperature  $T_1$  is diminished and the total temperature held constant

$$\bar{\tau}_{p_1}/p_t = \left( \bar{\tau}_{T_1}/T_t \right)^{\frac{\gamma_i}{\gamma_i - 1}} e^{-w(T_t)} \quad (44)$$

so that

$$\tau_{p_1}/p_t = \left( \frac{\tau_{T_1}/T_t}{\bar{\tau}_{T_1}/T_t} \right)^{\frac{\gamma_i}{\gamma_i - 1}} e^{w(T_1)} \bar{\tau}_{p_1}/p_t \quad (45)$$

The fractional error defined by

$$f_{\tau p} \equiv \frac{\tau_{p_1}/p_t - \bar{\tau}_{p_1}/p_t}{\bar{\tau}_{p_1}/p_t}$$

can be written

$$f_{\tau p} = \left( \frac{\gamma_i}{\gamma_1} - 1 \right) \frac{\gamma_i}{\gamma_i - 1}$$

so that the value of the specific heat ratio corresponding to a given fractional error in  $\tau_{p_1}/p_t$  resulting from using the simplified result of equation (44) is given by

$$\gamma_1 = \frac{\gamma_i}{1 + \left( \frac{\gamma_i - 1}{\gamma_i} \right) f_{\tau p}} \quad (46)$$

Plane shock flows.- Two levels of simplification to the plane shock relations are considered. The first of these eliminates the need for the iterative solution across the shock wave. The second involves obtaining the equations for the asymptotic values of the real-gas corrections approached as the static temperature is reduced while the total temperature is held constant.

The equations for the plane shock flows are written in terms of the function  $\eta$  defined by equation (22). Evaluation of this function requires a knowledge of the static temperature downstream of the shock wave. This temperature is not known a priori so that an iterative solution of equation (26) for  $T_2/T_1$  is required in order to establish a value for  $T_2$  and subsequently  $\eta$ . This difficulty can be circumvented, with only small loss in accuracy, by recognizing that  $T_2$  is never more than a few percent different from the total temperature and that the  $\eta$  function is not very sensitive to errors in  $T_2$  of this order. Because of this,  $T_2$  can be replaced by  $T_t$  in the equation for  $\eta$ . A comparison of exact results for the real-gas correction to the static temperature ratio across a normal shock wave in pure  $\text{CO}_2$  to those obtained using the simplified  $\eta$  function is made in figure 5. The differences between the exact and the approximate results are shown to be small.

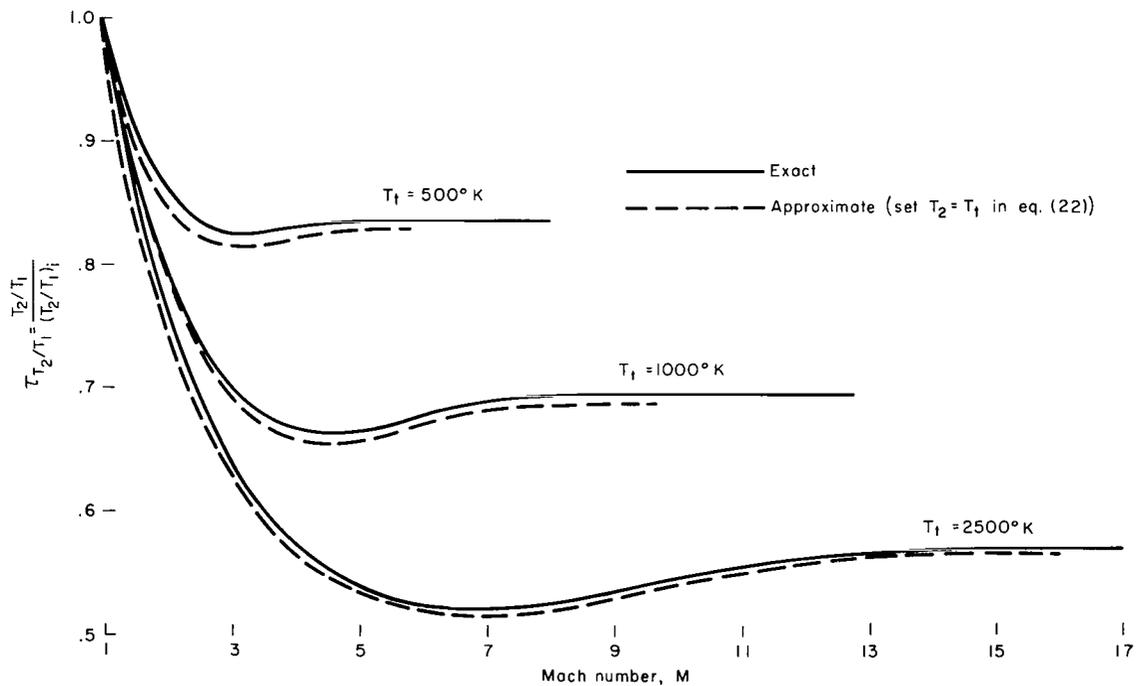


Figure 5.- Comparisons of exact and approximate results for the real gas correction to the static temperature ratio across a normal shock in carbon dioxide.

The results of figure 5 also show that the real-gas correction to the static temperature ratio approaches an asymptote as Mach number is increased. This behavior is typical of all of the real-gas corrections to the shock

parameters as it is for the corrections to the isentropic flow quantities. Equations for asymptotic values of all the functions can be readily obtained after finding the form of the  $\eta$  function for  $T_1 \rightarrow 0$  and  $T_2 = T_t$ . Under these circumstances,  $\gamma_1 \rightarrow \gamma_i$  so that

$$\bar{\eta} = \gamma_i^2 M_1^4 \left[ \gamma_i^2 - \frac{(\gamma_i^2 - 1)}{\bar{\tau}_{T_1/T_t}} \right]$$

Substituting this relation for  $\bar{\eta}$  into equations (23), (25), (26), and (29) and by dropping terms of order  $1/\gamma_1 M_1^2 = RT_1/V_1^2$  gives the following equations for the asymptotic values of density, pressure, temperature, and speed ratios, respectively.

$$\bar{\tau}_{\rho_2/\rho_1} = \frac{(\gamma_i - 1)}{\gamma_i - \sqrt{\gamma_i^2 - \frac{(\gamma_i^2 - 1)}{\bar{\tau}_{T_1/T_t}}}} \quad (47)$$

$$\bar{\tau}_{p_2/p_1} = \frac{1}{2} \left[ 1 + \sqrt{\gamma_i^2 - \frac{(\gamma_i^2 - 1)}{\bar{\tau}_{T_1/T_t}}} \right] \quad (48)$$

$$\bar{\tau}_{T_2/T_1} = \frac{1}{2} \left[ \frac{(\gamma_i + 1)}{\bar{\tau}_{T_1/T_t}} - \gamma_i + \sqrt{\gamma_i^2 - \frac{(\gamma_i^2 - 1)}{\bar{\tau}_{T_1/T_t}}} \right] \quad (49)$$

$$\bar{\tau}_{V_2/V_1} = \frac{1}{(\gamma_i - 1)} \left[ \gamma_i - \sqrt{\gamma_i^2 - \frac{(\gamma_i^2 - 1)}{\bar{\tau}_{T_1/T_t}}} \right] \quad (50)$$

The asymptotic value of the real-gas correction to the sound speed ratio is obtained from equation (30). The result is

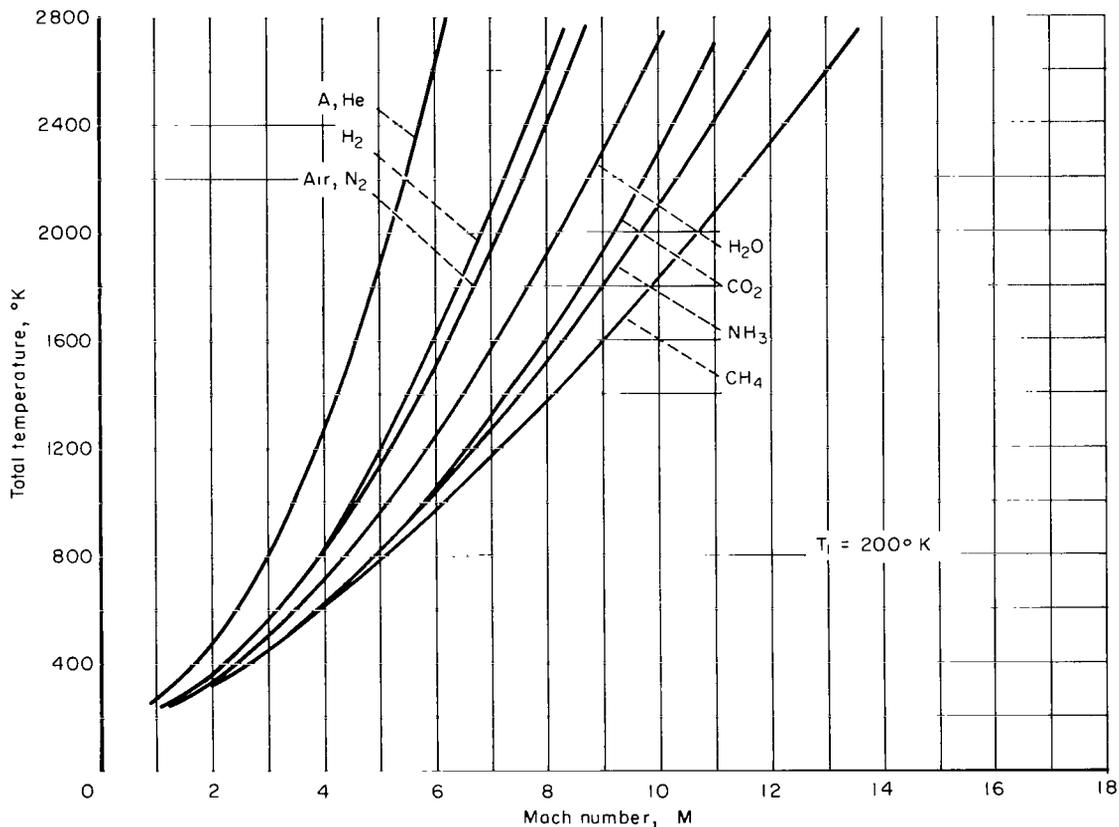
$$\bar{\tau}_{a_2/a_1} = \sqrt{\bar{\tau}_{T_2/T_1} \frac{\gamma_t}{\gamma_i}} \quad (51)$$

Equations for the asymptotic values of the corrections to the remaining parameters are given by combinations of equations (47) through (51).

### DISCUSSION

Using the equations developed in this study it is a simple matter to determine the effects of changing gas composition on quantities of interest in isentropic and plane shock flows. Some sample comparisons of the behavior of a number of gases will be made and discussed.

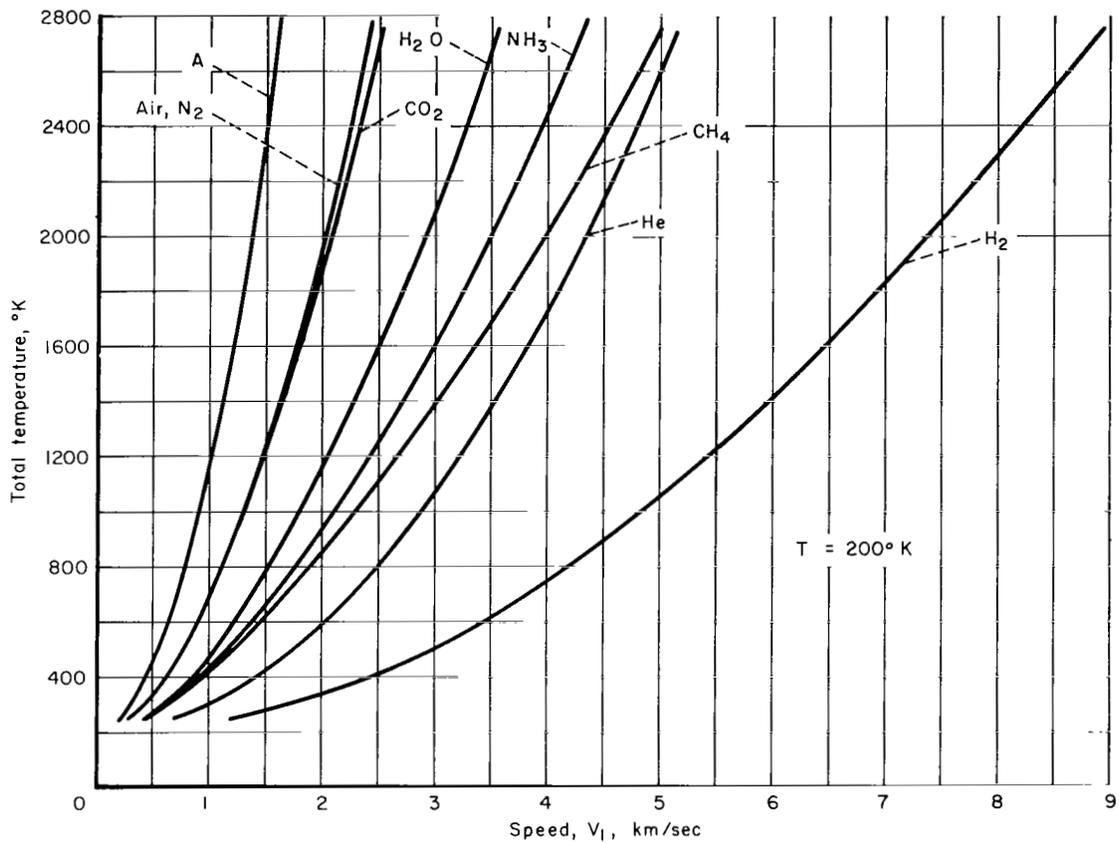
Consider first the situation of a body flying in an atmosphere. The variation of total temperature with flight Mach number and flight speed for a number of pure gases is shown in figure 6. These results were obtained by means of equations (12), (13), and (14) and are for an arbitrarily chosen ambient temperature of  $200^{\circ}\text{K}$ . It is shown in figure 6(a) that changes in



(a) Variation of total temperature with Mach number.

Figure 6.- Effect of gas composition on the variation of total temperature with Mach number and speed;  $T_1 = 200^{\circ}\text{K}$ .

gas composition have large effects on total temperature when results are compared at the same Mach number. However, as shown in figure 6(b), the comparisons of results for the various gases are considerably different when made



(b) Variation of total temperature with speed.

Figure 6.- Concluded.

at the same speed. For example, the difference between the total temperature in carbon dioxide and that in nitrogen at comparable Mach numbers grows with increasing Mach number to exceed  $800^\circ K$  while the difference never exceeds about  $150^\circ K$  when evaluated at comparable speeds. In contrast to this result the differences between the curves for helium and air are larger when compared at the same speed than when compared at the same Mach number. In addition to magnitudes, the algebraic sign of the difference is not always the same for the results presented as a function of Mach number as for the results presented as a function of speed. Considerations of this sort indicate that when comparing aerodynamic quantities for various gas compositions it will be necessary to exercise care in drawing conclusions since the conclusions can be strongly dependent upon the basis of the comparisons.

As a second illustration of the aerodynamic behavior of various gases predictable by the equations derived in this analysis, consider an application

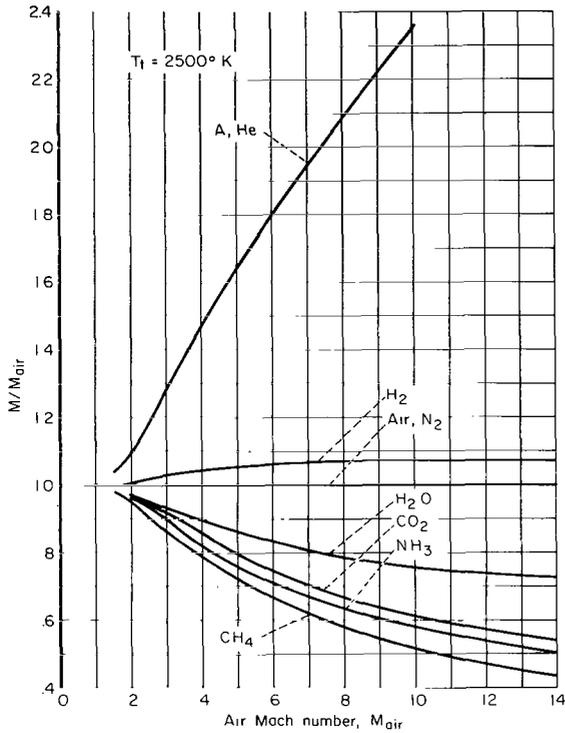


Figure 7.- Effect of gas composition on Mach number obtainable in a wind tunnel.

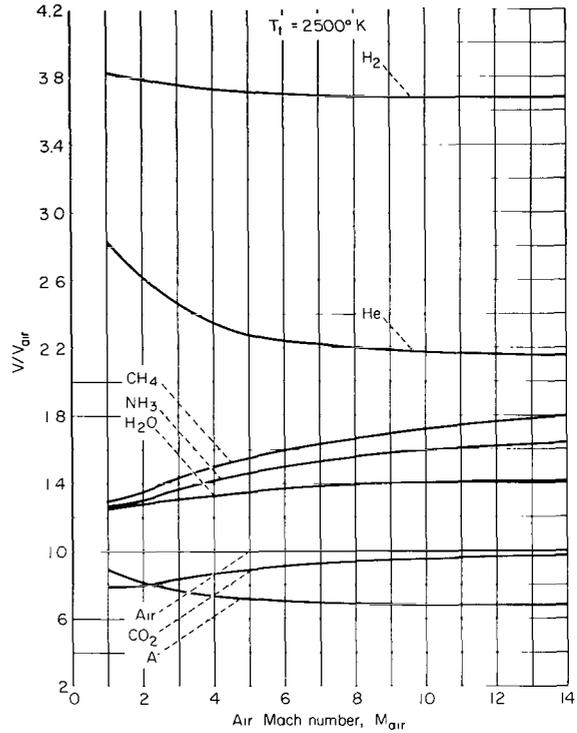


Figure 8.- Effect of gas composition on speed obtainable in a wind tunnel.

to wind-tunnel flows. In many wind tunnels the total or reservoir temperature is held constant during a test as opposed to the flight case where the ambient temperature is fixed. For a total temperature of  $2500^{\circ}$  K, the ratio of flow Mach number obtainable in each of a number of gases to Mach number obtainable in air is presented in figure 7 as a function of air Mach number. Results presented in this form readily show estimates of the performance capabilities of presently operating air facilities when operated with planetary gas mixtures. It is apparent from the results of figure 7 that the flow Mach number obtainable in a given nozzle is strongly dependent on gas composition. A nozzle designed to produce a given Mach number in air will produce higher Mach number flows of gases having a smaller ratio of internal energy to total energy than air and lower Mach number flows of gases having a larger ratio of internal energy to total energy than air for a given reservoir temperature.

The ratio of wind-tunnel flow speeds obtainable with each of a number of gases to that obtainable with air is shown in figure 8 as a function of air Mach number for a total temperature of  $2500^{\circ}$  K. The relationships between the various gases in terms of flow speed obtainable in a wind tunnel designed for a given Mach number in air are quite different from those shown in figure 7 in terms of Mach number. All the differences between the comparisons illustrated in figure 7 and those shown in figure 8 are due to differences in sound speed in the various gases.

As a final comparison of the aerodynamic behavior of a number of possible constituents of planetary atmospheres, consider the stagnation pressure coefficient. This quantity is of particular interest since a number of theories for predicting pressures and forces on entry bodies relate the predictions to the stagnation point pressure. The stagnation point pressure coefficients for a number of gases normalized to that for real air are presented as a function of flight speed in figure 9. It is interesting to note that while many of the

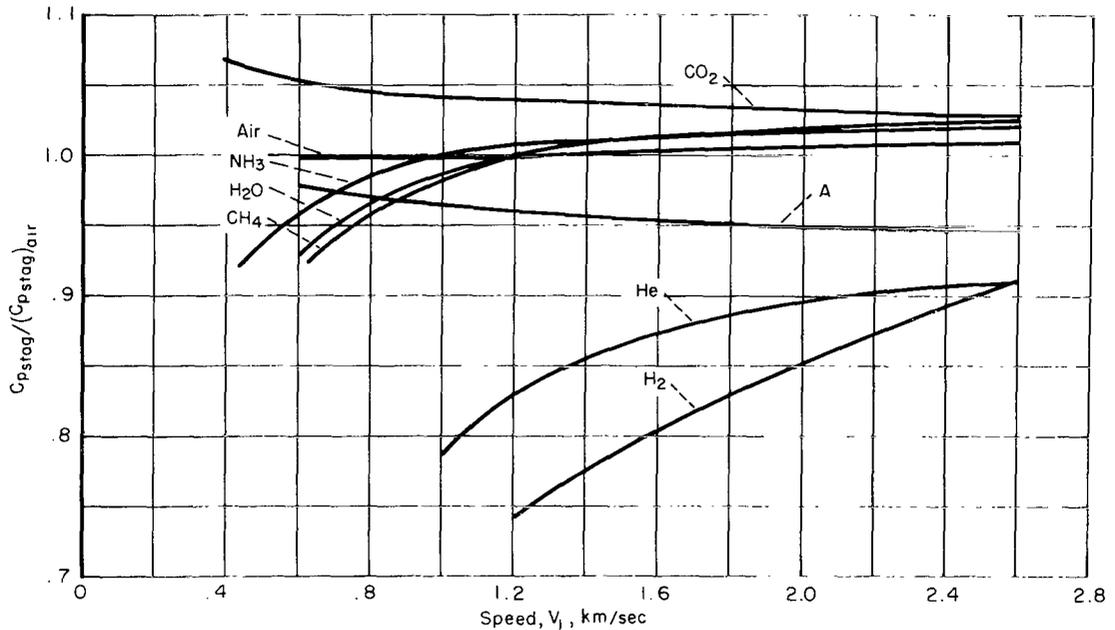


Figure 9.- Effect of gas composition on stagnation point pressure coefficient;  $T_1 = 200^\circ \text{K}$ .

quantities characterizing aerodynamic flows are very sensitive to gas composition, the stagnation pressure coefficients for a number of gases differ by only a few percent from one gas to another.

#### CONCLUDING REMARKS

Equations for aerodynamic quantities of interest in isentropic one-dimensional flows and for properties across plane shock waves in arbitrary mixtures of undissociated planetary gases have been derived. It was shown that the equations could be cast in forms not prohibitively complicated by making the following principal assumptions for the gas model: (a) all gas components of a mixture are thermally perfect so as to be governed by the perfect gas law  $p = \rho RT$ ; (b) molecular vibrations behave like harmonic

oscillators with no coupling between rotational and vibrational degrees of freedom; and (c) electronic energy is negligible compared to the total energy. Simplified equations giving approximate results over more limited ranges of the variables were also obtained along with estimates of the error introduced by the approximations.

Various uses for the equations were demonstrated by making sample comparisons of the aerodynamic behavior of a number of gases. In one comparison, estimates of the flow Mach number and speed capabilities of a wind tunnel operated with a number of pure gases in terms of its performance in air were presented. It was shown for the same reservoir temperature that a nozzle designed to produce a given Mach number in air will produce higher Mach number flows of gases having a smaller ratio of internal energy to total energy, at a given temperature, than air and lower Mach number flows of gases having a larger ratio of internal energy to total energy, at a given temperature, than air. It is also shown that while some of the quantities characterizing aerodynamic flows are very sensitive to gas composition, the stagnation pressure coefficients for a number of gases differ by only a few percent from one gas to another.

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National Aeronautics and Space Administration  
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TABLE I.- CONSTANTS FOR SELECTED GASES

(a) Linear molecule gases

Gas	Molecular weight	Vibrational energy constants, $\theta_n$ , $^{\circ}\text{K}$ (1)
CO	28	3121.85
CO <sub>2</sub>	44	959.9(2), 1928, 3379.5
H <sub>2</sub>	2	6322.5
NO	30	2738.44
N <sub>2</sub>	28	3394.3
N <sub>2</sub> O	44	845.8(2), 1853.1, 3218
O <sub>2</sub>	32	2273.35
air	29	3055.5

(b) Nonlinear molecule gases

Gas	Molecular weight	Vibrational energy constants, $\theta_n$ , $^{\circ}\text{K}$ (1)
CH <sub>4</sub>	16	1879(3), 2195(2), 4192.1, 4344.7(3)
H <sub>2</sub> O	18	2294.4, 5253, 5402.7
NH <sub>3</sub>	17	1340.08, 1392.58, 2341.6, 4798.7, 4801, 4911

<sup>1</sup>Number in parentheses denotes the number of degrees of freedom to which the constant it follows applies.

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